

THE ELASTIC PROPERTIES

OF

HELICAL SPRINGS

BY

JOSEPH WARREN MILLER, JR., B.S., A.M.

A DISSERTATION

SUBMITTED IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE

DEGREE OF DOCTOR OF PHILOSOPHY IN THE FACULTY OF

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THE ELASTIC PROPERTIES OF HELICAL SPRINGS.

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1. *Historical Note.*—In 1814 M. J. Binet¹ noticed the fundamental principle that helical springs act chiefly by torsion. Apparently nothing was done in this line between that time and 1848, when James Thomson² showed that a helical spring, of infinitely small slope, acts exactly as a torsion balance using the same wire straightened. He verified his theory by experiments. The fact that a spring acts as a torsion balance means it follows Hooke's law. It is this close accord with Hooke's law that is utilized in the ordinary spring balance. Apparently a complete statement of the theory of helical springs was first given by Kelvin and Tait in their "Natural Philosophy,"³ 1867. Certain aspects of the theory were also given by Clebsch⁴ and Kirchhoff.⁵ Some of the most recent contributions upon this subject are those of Prof. A. E. H. Love, who devotes several pages of his "Theory of Elasticity,"⁶ to helical springs, and who also published a paper on "The Propagation of Waves of Elastic Displacement along a Helical Wire," which appeared in the volume of "Memoirs on the occasion of the jubilee of Sir George G. Stokes."⁷

¹ Journal de L'École Royal Polytechnique, Tome X., p. 419.

² Camb. and Dub. Math. Journal, Vol. III., p. 258.

³ Vol. II., Arts. 604-607.

⁴ Vorlesungen über Mathematische Physik, Mechanik, 1876.

⁵ Theorie der Elasticität fester Körper, 1864.

⁶ Vol. II., Chapters XIV.-XVI., 1892.

⁷ p. 346, 1900.

The purpose of the present paper is to set forth the results of a research conducted in the Department of Mechanics in Columbia University. The original object of this enquiry was to learn whether helical springs could be used to determine the acceleration due to gravity. Early experiments showed the need of a more extended enquiry into the behavior of springs under static conditions. Thus the following investigation refers chiefly to the static constants and properties of a set of helical springs of different dimensions and metals.

2. *General Theory.*—To define the bending and twisting essential in a wire whose elastic central line assumes the form of a helix of slope α on a cylinder of radius r , consider the relations shown in the accompanying diagram.

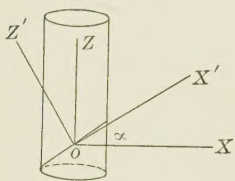


Fig. 1.

At any point O on the elastic central line draw in the plane tangent to the cylinder the tangent to the helix OX' , the element OZ , the horizontal line OX , and the line OZ' perpendicular to OX' . Let ds be an element of the elastic central line measured from O towards X' . The angular rotation about the axis of the cylinder corresponding to this element, or what is the same thing, the angle between two meridian planes through the ends of ds , is $ds \cos \alpha/r$.

This rotation is equivalent to its components about OZ' and OX' that is to $ds \cos \alpha \cos \alpha/r$ and $ds \cos \alpha \sin \alpha/r$. The first of these is a pure bending and the second is a pure twisting when applied to the wire. Thus for a right cylindrical helix the amounts of bending and twisting per unit length of the wire are $\cos^2 \alpha/r$ and $\cos \alpha \sin \alpha/r$ respectively.¹ The potential energy of a wire of length s , measured along its elastic central line, bent into a helix of slope α and radius r is hence easily deduced. For the part due to bending is proportional to $(\cos^2 \alpha/r)^2$ per unit length, and the part due to twisting is proportional to $(\cos \alpha \sin \alpha/r)^2$ per unit length. Therefore the work done in producing the helix is

$$\frac{1}{2} s \left\{ A \frac{\cos^2 \alpha \sin^2 \alpha}{r^2} + B \frac{\cos^4 \alpha}{r^2} \right\},$$

¹ Compare Thompson and Tait's *Natural Philosophy*, Part II., Art. 602, p. 137.

where A is the modulus of twist, and B is the modulus of bending.

If we write

$$\begin{aligned} z &= s \sin \alpha, & \varphi r &= s \cos \alpha, \\ z_0 &= s \sin \alpha_0, & \varphi_0 r_0 &= s \cos \alpha_0; \end{aligned} \quad (1)$$

where z_0 is the axial length of the helix, r_0 is the radius of the cylinder containing the elastic central line of the wire, and φ_0 the angle between two planes through the ends of the helix and its axis in the unstrained condition; and, similarly, z , r , and φ the corresponding quantities for the strained condition, then the curvature

$$\cos^2 \alpha / r = \varphi \cos \alpha / s$$

and the change in curvature, per unit length, is

$$(\varphi \cos \alpha - \varphi_0 \cos \alpha_0) / s.$$

The potential energy per unit length, due to the bending, is proportional to $(\varphi \cos \alpha - \varphi_0 \cos \alpha_0)^2 / s^2$, and hence this part of the energy for the whole wire is

$$\frac{1}{2} \frac{B}{s} (\varphi \cos \alpha - \varphi_0 \cos \alpha_0)^2.$$

Similarly the change in twist per unit length is

$$\frac{\cos \alpha \sin \alpha}{r} - \frac{\cos \alpha_0 \sin \alpha_0}{r_0} = (\varphi \sin \alpha - \varphi_0 \sin \alpha_0) / s;$$

whence the potential energy due to twist alone is

$$\frac{1}{2} \frac{A}{s} (\varphi \sin \alpha - \varphi_0 \sin \alpha_0)^2.$$

Calling the entire potential energy V , it follows that

$$V = \frac{1}{2} \frac{A}{s} (\varphi \sin \alpha - \varphi_0 \sin \alpha_0)^2 + \frac{1}{2} \frac{B}{s} (\varphi \cos \alpha - \varphi_0 \cos \alpha_0)^2. \quad (2)$$

Let

P = longitudinal pull

and

L = torsional moment.

Then we have

$$P = \frac{dV}{dz} = \frac{dV da}{da dz} = \frac{1}{s \cos a} \frac{dV}{da},$$

or

$$P = \frac{A}{s^2} (\varphi \sin a - \varphi_0 \sin a_0) \varphi + \frac{B}{s^2} (\varphi_0 \cos a_0 - \varphi \cos a) \varphi \tan a, \quad (3)$$

and

$$L = \frac{dV}{d\varphi} = \frac{A}{s} (\varphi \sin a - \varphi_0 \sin a_0) \sin a + \frac{B}{s} (\varphi \cos a - \varphi_0 \cos a_0) \cos a. \quad (4)$$

If the applied couple L is zero equation (4) gives

$$\frac{A}{B} = \frac{\cos a}{\sin a} \cdot \frac{\varphi_0 \cos a_0 - \varphi \cos a}{\varphi \sin a - \varphi_0 \sin a_0}. \quad (5)$$

If we introduce (5) in (3) there results

$$\frac{Ps^2}{A} = (\varphi \sin a - \varphi_0 \sin a_0) \varphi \sec^2 a. \quad (6)$$

Expanding the second member of this by Maclaurin's series and solving for P we get

$$P = \frac{A\varphi_0^2}{s^2} \sec^2 a \left\{ \sin a - \sin a_0 + (2 \sin a - \sin a_0) \frac{\Delta\varphi}{\varphi_0} + \sin a \left(\frac{\Delta\varphi}{\varphi_0} \right)^2 \right\}, \quad (7)$$

$$= \frac{A\varphi_0^2}{s^3} \sec^2 a \left\{ z - z_0 + (2z - z_0) \frac{\Delta\varphi}{\varphi_0} + z \left(\frac{\Delta\varphi}{\varphi_0} \right)^2 \right\}. \quad (8)$$

From (8) we see that P varies nearly as $(z - z_0) \sec^2 a$, since $\frac{\Delta\varphi}{\varphi_0}$ is generally small. If we divide (3) by B and then introduce (5) we get

$$\frac{Ps^2}{B} = (\varphi_0 \cos a_0 - \varphi \cos a) \varphi (\tan a + \cot a), \quad (9)$$

$$= (\varphi_0 \cos a_0 - \varphi \cos a) 2\varphi / \sin 2a.$$

As before, expanding the second member by Maclaurin's series,

$$P = \frac{2B\varphi_0^2}{s^2 \sin 2a} \left\{ \cos a - \cos a_0 + (\cos a_0 - 2 \cos a) \frac{\Delta\varphi}{\varphi_0} - \cos a \left(\frac{\Delta\varphi}{\varphi_0} \right)^2 \right\}. \quad (10)$$

From (7) and (10) we have

$$\frac{1}{A} = \frac{\varphi_0^2 \sec^2 a}{Ps^2} \left\{ \sin a - \sin a_0 + (2 \sin a - \sin a_0) \frac{\Delta\varphi}{\varphi_0} + \sin a \left(\frac{\Delta\varphi}{\varphi_0} \right)^2 \right\}, \quad (11)$$

$$\frac{1}{B} = \frac{2\varphi_0^2}{Ps^2 \sin 2a} \left\{ \cos a_0 - \cos a - (2 \cos a - \cos a_0) \frac{\Delta\varphi}{\varphi_0} - \cos a \left(\frac{\Delta\varphi}{\varphi_0} \right)^2 \right\}. \quad (12)$$

In case $\Delta\varphi$ is negligible these formulas give A and B independently. In general, however, terms in $\Delta\varphi$ may not be neglected in (12). For springs of small slope (11) will usually give a close approximation to A . Either formula may be used for exact computations when $\Delta\varphi$ or A/B is known, as shown below.

From (5) we get

$$\begin{aligned} \frac{\varphi}{\varphi_0} &= \frac{B \cos a_0 + A \sin a_0 \tan a}{B \cos a + A \sin a \tan a} = \frac{B \cos a \cos a_0 + A \sin a \sin a_0}{B \cos^2 a + A \sin^2 a} \\ &= \frac{\cos(a - a_0) + x \cos(a + a_0)}{1 + x \cos 2a}, \end{aligned} \quad (13)$$

wherein

$$x = \frac{B - A}{B + A}.$$

To compute the change in φ the following formula is best

$$\begin{aligned} \frac{\varphi}{\varphi_0} - 1 &= \frac{\varphi - \varphi_0}{\varphi_0} = \frac{\Delta\varphi}{\varphi_0} \\ &= 2 \sin \frac{1}{2}(\alpha - \alpha_0) \frac{x \sin \frac{1}{2}(3\alpha + \alpha_0) - \sin \frac{1}{2}(\alpha - \alpha_0)}{1 + x \cos 2\alpha}. \end{aligned} \quad (14)$$

From (14) we see that $\Delta\varphi = 0$ for two values of α , namely, for $\alpha = \alpha_0$ and for the value obtained from

$$x \sin \frac{1}{2}(3\alpha + \alpha_0) = \sin \frac{1}{2}(\alpha - \alpha_0),$$

and, approximately, from

$$\alpha - \alpha_0 = x \sin 2\alpha_0 + \frac{3}{2} x^2 \sin 4\alpha_0 + \dots$$

From (14) we also see that as the spring elongates $\varphi - \varphi_0$ increases up to a maximum and then decreases. If we write $\xi = A/B$, then $x = (1 - \xi)/(1 + \xi)$, and the maximum of φ or $\Delta\varphi$ is given by the following cubic equation in $\tan \alpha$

$$\tan \alpha = \frac{\xi}{2\xi - 1} \tan \alpha_0 \frac{1 + (2 - \xi) \tan^2 \alpha}{\xi + \frac{\xi}{2\xi - 1} \tan^2 \alpha}. \quad (15)$$

Since, for the case here considered, $\frac{2}{3} < \xi < 1$, as shown below by equation (21),

$$2 - \xi < \frac{\xi}{2\xi - 1},$$

a real root of the above cubic lies between

$$(2 - \xi) \tan \alpha_0 \quad \text{and} \quad \frac{\xi}{2\xi - 1} \tan \alpha.$$

From the graph of (15) we see that there is but one real root.¹

If the elongation $z - z_0$ of maximum twist be measured ξ comes readily by successive approximations from

$$\frac{2\xi - 1}{\xi} = \frac{\tan \alpha_0}{\tan \alpha} + (2 - \xi) \tan \alpha \tan \alpha_0 - \tan^2 \alpha_0. \quad (16)$$

¹Supposing $\xi = \frac{8}{10}$, its value when Poisson's ratio is $\frac{1}{4}$, the real root lies between $\frac{6}{5} \tan \alpha_0$ and $\frac{4}{3} \tan \alpha_0$.

Also from (5), if $\varphi = \varphi_0$,

$$\frac{A}{B} = \xi = \frac{\cos \alpha}{\sin \alpha} \cdot \frac{\cos \alpha_0 - \cos \alpha}{\sin \alpha - \sin \alpha_0} = \frac{\tan \frac{1}{2}(\alpha + \alpha_0)}{\tan \alpha}. \quad (17)$$

From A , B and $\frac{A}{B}$ we get

E = Young's modulus,

n = rigidity, or slide modulus,

and

η = Poisson's ratio = ratio of lateral contraction to longitudinal extension.

Thus, if a is the radius of the cross section of the wire,¹

$$A = \frac{1}{2}\pi a^4 n, \quad \text{whence} \quad n = \frac{2A}{\pi a^4}; \quad (18)$$

$$B = \frac{1}{4}\pi a^4 E, \quad \text{whence} \quad E = \frac{4B}{\pi a^4}; \quad (19)$$

$$2n(1 + \eta) = E, \quad \text{whence} \quad \eta = \frac{B}{A} - 1; \quad (20)$$

and

$$0 < \eta < \frac{1}{2}, \quad \text{whence} \quad \frac{2}{3} < \xi < 1. \quad (21)$$

3. *Methods of Observing and Apparatus.*—Most of the commercial helical springs are “closed”; that is, the coils of wire are in close contact and sometimes require considerable force to separate them. Before “opening,” that is, before giving the spring a permanent set which leaves the coils open, or apart, when hanging under no load, the radius of the cross section of the wire, a , can be found by measuring, with an ordinary standard scale, the lengths of several different sets (in hundreds) of coils. The radius of the cylinder, r_0 , which envelopes the elastic central line of the wire, when hanging under no load, can be found if we know the external diam-

¹ The modulus of twist is equal to the modulus of rigidity times the polar moment of inertia of the circular cross section of the wire; and the modulus of bending is equal to Young's modulus (or the engineer's modulus) times the moment of inertia of the same cross section with respect to its diameter. For the relation between E , n and η see, for example, section 3 of Saint-Venant's edition of Clebsch, *Théorie de l'Élasticité des Corps Solides*, Paris, 1883.

eter of the helix and the radius of the wire. The external diameter of the helix was measured with a Brown and Sharpe micrometer screw-gauge. Thus

$$r_0 = \frac{1}{2} (\text{external diameter of helix} - \text{diameter of wire}).$$

In order to avoid the uncertainty as to where the helix began and ended small dots of india ink were placed on the wire nearly in the same vertical plane, one several turns from the top and one several turns from the bottom. The number of turns from one dot to the other multiplied by 2π gives φ_0 . The ends of all the springs were annealed and drawn out straight, thus bringing these ends into the axis of the cylinder of the helix. The method of suspension was to clamp the straight part of the wire in a vise which was fastened to a stable wall. By means of the micrometers of the telescopes of a Geneva cathetometer of precision used in the work it was possible to determine the exact positions of the dots with respect to one another, and thus obtain a correction to the value of φ_0 mentioned above due to the fact that the dots were not usually quite in the same vertical plane.

The vertical distance between the dots was determined by setting the movable parallel wires of the cathetometer telescopes on the upper and lower dots respectively and then turning the shaft, upon which the telescopes are mounted, until a standard meter of white bronze, which was mounted beside the spring, came into view. This meter is divided into millimeters and was made vertical by means of a plumb line. Although we do not know the absolute length of this meter an examination has shown that there are no graduation errors large enough to affect the work here considered.

A hollow brass cylinder with a three-jaw chuck enabled me to apply any load P up to six kilograms. Shot of small size were used to secure loads of the desired amounts.

The method of finding the position of maximum twist was as follows: On the straightened part of the lower end of the helix a small cork was slipped. Through the cork a straight piece of fine wire was thrust, thus forming an index. The cork was twisted until the index pointed to a plumb line suspended nearby. The empty cylinder was then clamped on the wire protruding through

the cork and in every case the index indicated a twist in accord with the law defined by equation (14). The index was then moved back until it again pointed to the plumb-line. The cylinder was then gradually loaded with shot, the index meanwhile being turned back successively to the plumb line. The rate of twist is quite rapid at first but as the position of maximum twist is approached the rate becomes less and less. The amount of twist was small in every instance, in no case did it exceed 40° . However, a considerable error in the elongation of maximum twist produces only a slight error in the value of ξ as shown by equation (16).

To find the position of no twist the index is not moved from its original position, in which it points to the plumb line, but shot are added until the index again points to the plumb line.

Thus, knowing φ_0 , r_0 , z_0 , and z , we find α_0 , s , and a from the following formulas :

$$\begin{aligned}\tan \alpha_0 &= z_0/\varphi_0 r_0, \quad s = z_0/\sin \alpha_0, \\ \sin a &= z/s, \quad s = \varphi_0 r_0/\cos \alpha_0.\end{aligned}\tag{22}$$

4. *Specimen of Computations of Constants of a Spring.*—For a copper-plated steel spring the following data were observed: diameter of wire 0.1223 cm., external diameter of helix 0.8515 cm., number of turns 295, mean axial length of helix under no load 39.615 cm., axial length at position of maximum twist 54.250 cm., axial length at position of zero twist 68.955 cm., produced by load $P = 5495.9$ grams.

(a) Computation of r_0 , α_0 and s , with check for s .

$$\begin{aligned}r_0 &= \frac{1}{2}(0.8515 - 0.1223) = 0.3646 \text{ cm.} \\ \text{Log } 295 &= 2.46982 \\ \text{Log } 2\pi &= 0.79818 \\ \text{Log } \varphi_0 &= 3.26800 \quad \varphi_0 = 1853.5 \\ \text{Log } r_0 &= 9.56182 \\ \text{Log } \varphi_0 r_0 &= 2.82982 \\ \text{Log } z_0 &= 1.59786 \\ \text{Log } \tan \alpha_0 &= 8.76804 \quad \alpha_0 = 3^\circ 21' 17.3'' \\ \text{Log } \sin \alpha_0 &= 8.76730 \\ \text{Log } z_0 &= 1.59786\end{aligned}$$

$$\text{Log } s = 2.83056 \quad s = 676.96$$

$$\text{Log } \cos \alpha_0 = 9.99926$$

$$\text{Log } \varphi_0 r_0 = 2.82982$$

$$\text{Log } s = 2.83056 \quad s = 676.96$$

(k) Computation of α , $\xi = A/B$ and η from position of zero twist and maximum twist.

$$\text{Log } 68.955 = 1.83857$$

$$\text{Log } s = 2.83056$$

$$\text{Log } \sin \alpha = 9.00801$$

$$\alpha = 5^\circ 50' 47.0''$$

$$\alpha_0 = 3 \quad 21 \quad 17.3$$

$$\alpha + \alpha_0 = 9 \quad 12 \quad 04.3$$

$$\alpha - \alpha_0 = 2 \quad 29 \quad 29.7$$

$$\frac{1}{2}(\alpha + \alpha_0) = 4 \quad 36 \quad 02.2$$

$$\frac{1}{2}(\alpha - \alpha_0) = 1 \quad 14 \quad 44.8$$

$$\text{Log } \tan \frac{1}{2}(\alpha + \alpha_0) = 8.90563$$

$$\text{Log } \tan \alpha = 9.01028$$

$$\text{Log } \xi = A/B = 9.89535 \quad A/B = 0.7859$$

$$\text{Log } 54.250 = 1.73440$$

$$\text{Log } s = 2.83056$$

$$\text{Log } \sin \alpha = 8.90384 \quad \alpha = 4^\circ 35' 47.9''$$

$$\text{Log } \tan \alpha = 8.90524$$

$$\text{Log } \tan^2 \alpha = 7.81048 \quad \tan^2 \alpha = 0.00646$$

$$\text{Log } \tan \alpha_0 = 8.76804$$

$$\text{Log } \tan \alpha = 8.90524$$

$$\text{Log } \tan \alpha_0 / \tan \alpha = 9.86280 \quad \tan \alpha_0 / \tan \alpha = 0.72912.$$

$$\text{Log } \tan \alpha \tan \alpha_0 = 7.67328$$

$$\text{Log } (2 - \xi) = 0.08408$$

$$\text{Log } (2 - \xi) \tan \alpha \tan \alpha_0 = 7.75736 \quad (2 - \xi) \tan \alpha \tan \alpha_0 = 0.00572.$$

$$0.72912 - 0.00646 + 0.00572 = 0.72838.$$

Therefore

$$2\xi - 1 = 0.72838 \xi,$$

or

$$1.2716 \xi = 1.$$

Now

$$1/\xi - 1 = B/A - 1 = \eta = 0.2716$$

and

$$\xi = A/B = 0.7864.$$

(c) Computation of A .

$$\begin{array}{ll} \text{Log } 2 & = 0.30103 \\ \text{Log } \sin \frac{1}{2}(a - a_0) & = 8.33728 \\ \text{Log } \cos \frac{1}{2}(a + a_0) & = 9.99860 \end{array} \left. \vphantom{\begin{array}{l} \text{Log } 2 \\ \text{Log } \sin \frac{1}{2}(a - a_0) \\ \text{Log } \cos \frac{1}{2}(a + a_0) \end{array}} \right\} \text{Log } (\sin a - \sin a_0). \\ \text{Log } \sec^2 a & = 0.00453 \\ \text{Log } \varphi_0^2/s^2 & = 0.87488 \\ \text{colog } P & = 6.25996 \\ \text{Log } 1/A & = 5.77628 \quad 1/A = 0.000059742. \\ \text{Log } A & = 4.22372 \quad A = 16739.$$

(d) Computation of B , E and n .

$$\begin{array}{ll} \text{Log } 1/A = 5.77634 & (\text{Mean of five measures.}) \\ \text{Log } A/B = 9.89570 \\ \text{Log } 1/B = 5.67204 \end{array}$$

$$\begin{array}{ll} \text{Log } B = 4.32796 & B = 21280. \\ \text{Log } 4 = 0.60206 \\ \text{Log } 4B = 4.93002 \\ \text{Log } \pi a^4 = 5.64274 \\ \text{Log } E = 9.28728 & E = 1937700000. \\ \text{Log } E = 9.28728 & n = \frac{1}{2} \xi E. \\ \text{Log } \xi = 9.89570 \\ \text{Log } \xi E = 9.18298 \\ \text{Log } 2 = 0.30103 \\ \text{Log } n = 8.88195 & n = 762000000. \end{array}$$

(ε) Computation of $\frac{\Delta\varphi}{\varphi_0}$ at position of maximum twist.

$$x = (1 - \xi)/(1 + \xi) \quad \xi = 0.7865.$$

$$\text{Log } (1 - \xi) = 9.32940$$

$$\text{Log } (1 + \xi) = \underline{0.25300}$$

$$\text{Log } x = 9.07740$$

$$\frac{1}{2}(3\alpha + \alpha_0) = 8^\circ 34' 20''.5$$

$$\frac{1}{2}(\alpha - \alpha_0) = 0^\circ 37' 15''.3$$

$$2\alpha = 9^\circ 11' 35''.8$$

$$\text{Log } x = 9.07740$$

$$\text{Log } \sin \frac{1}{2}(3\alpha + \alpha_0) = \underline{9.17336}$$

$$\text{Log } x \sin \frac{1}{2}(3\alpha + \alpha_0) = 8.25076 \quad x \sin \frac{1}{2}(3\alpha + \alpha_0) = 0.017814$$

$$\text{Log } \sin \frac{1}{2}(\alpha - \alpha_0) = 8.03490 \quad \sin \frac{1}{2}(\alpha - \alpha_0) = 0.010836$$

$$x \sin \frac{1}{2}(3\alpha + \alpha_0) - \sin \frac{1}{2}(\alpha - \alpha_0) = 0.006978$$

$$\text{Log } \sin \frac{1}{2}(\alpha - \alpha_0) = 8.03490$$

$$\text{Log } 2 = 0.30103$$

$$\text{Log } 0.006978 = \underline{7.84373}$$

$$\text{Log numerator} = \underline{6.17966}$$

$$\text{Log } x = 9.07740$$

$$\text{Log } \cos 2\alpha = \underline{9.99438}$$

$$\text{Log } x \cos 2\alpha = 9.07178$$

$$1 + x \cos 2\alpha = 1.1180$$

$$\text{Log } (1 + x \cos 2\alpha) = \text{Log denominator} = 0.04844$$

$$\text{Log } \frac{\Delta\varphi}{\varphi_0} = 6.13122 \quad \frac{\Delta\varphi}{\varphi_0} = 0.00013528.$$

(f) Table of values for $\Delta\phi/\phi_0$ and $\Delta\phi$ for different values of α .

α	$\Delta\phi/\phi_0$	$\Delta\phi$
$\alpha_0 = 3^\circ 21' 17.3''$	+ 0.0000000	+ 0.0°
3 59 17.3	+ 0.00008635	+ 9.2
4 17 17.3	+ 0.00012701	+ 13.5
4 35 47.9	+ 0.00013528	+ 14.4
5 00 17.3	+ 0.00012047	+ 12.8
5 25 17.3	+ 0.00005711	+ 6.1
5 50 47.3	— 0.00002802	— 3.0
7 00 17.3	— 0.00037801	— 40.2

5. *Tables of Observed and Derived Results.*—The following tables give the numerical results appertaining to seven steel and three brass springs subjected to examination.

Table I. gives the observed data for these springs. The quantities measured are all linear. They are expressed by five significant figures; and it is believed that the precision of the measurements is correctly indicated by that number of figures.

Table II. gives the computed constants ϕ_0 , α_0 and s for the springs. As explained above (section 3), the value of ϕ_0 is 2π times the number of turns of the helix used, plus a small correction for the excess or defect of that number of turns from a round number. The precision of the data here given is also supposed to be represented without exaggeration by the figures used.

Table III. gives the observed and mean values of the modulus of twist A , and the ratio of the latter to the modulus of bending B or A/B . The probable errors attached to the mean values are derived from the discrepancies between the individual values and their mean for any spring, equal weights being assigned to the individual values. The values of A/B were derived mostly from the observed elongations of maximum twist ($\Delta\phi = a \text{ max.}$). A few values were derived from the observed elongations of zero twist ($\Delta\phi = 0$). Such values are designated by the letter b in parenthesis following them. Apparently, the results by these two methods are of about equal weight, and hence no distinction is made among them.

Table IV. gives the computed values of the modulus of bending B , the modulus of rigidity n , Young's modulus E , and Poisson's

ratio η . The values of E and n are given in dynes per square centimeter, the value of the normal acceleration at the place of observation, latitude $40^{\circ}48'28''$, being $g = 980.2^{cm}/(sec.)^2$.

TABLE I.

Observed Data of Springs.

Number and Metal of Springs.	Radius of Cross-Section of Wire. a	Radius of Unstrained Helix. r_0	Unstrained Length of Helix. z_0	Strained Length of Helix. z	Attached Mass. m
	Centimeter.	Centimeter.	Centimeter.	Centimeter.	Grams.
1. Copper-plated steel.	0.05680	0.32305	41.328	72.586	5408.8
				72.502	5390.2
				72.529	5392.9
				72.727	5427.4
				72.606	5403.3
				72.500	5385.9
2. Steel.	0.05605	0.32580	39.032	69.407	5756.2
				69.524	5773.6
				69.675	5802.0
				69.784	5818.5
				69.780	5816.2
3. Copper-plated steel.	0.06092	0.36430	38.153	44.273	1248.0
				46.731	1748.0
				49.174	2248.0
				51.612	2748.0
				53.593	3154.0
				56.030	3654.0
				56.049	3654.0
				60.949	4654.0
4. Copper-plated steel.	0.06115	0.36460	39.615	65.850	5654.0
				66.605	5804.0
				68.642	5444.8
				68.955	5495.9
				69.009	5504.2
5. Steel.	0.09325	1.09975	33.327	68.672	5438.9
				68.704	5440.9
				58.650	2378.1
				59.515	2459.0
				59.575	2466.0
				59.352	2453.5
				59.483	2456.8
				59.143	2442.0

TABLE I.—*Continued.**Observed Data of Springs.*

Number and Metal of Springs.	Radius of Cross-Section of Wire. a	Radius of Unstrained Helix. r_0	Unstrained Length of Helix. z_0	Strained Length of Helix. z	Attached Mass. m
	Centimeter.	Centimeter.	Centimeter.	Centimeter.	Grams.
6. Steel.	0.10328	1.10480	23.689	43.529	3384.3
				43.820	3402.2
				43.496	3348.3
				43.406	3332.0
				43.407	3333.3
7. Steel.	0.15315	1.8395	30.621	35.895	2070.0
				36.478	2298.0
				41.855	4368.1
				42.343	4452.0
8. Brass.	0.06526	0.4174	57.135	63.650	341.2
				80.890	1246.2
				82.753	1338.5
				88.940	1660.9
9. Brass.	0.08239	0.5716	58.050	63.000	341.2
				76.211	1246.2
				89.380	2135.5
				92.587	2347.8
10. Brass.	0.10280	0.8494	52.860	57.444	341.2
				70.030	1246.2
				71.106	1338.5
				84.245	2163.9

TABLE II.

Computed Constants of Springs.

Number of Spring.	Angle Between Ends of Spring. ϕ_0	Slope of Unstrained Spring. α_0	Length of Spring Along Elastic Central Line s .
	Radians.	° / "	Centimeter.
1	2054.6	3 33 46.5	665.03
2	1847.3	3 42 38.2	603.12
3	1690.2	3 32 43.0	616.99
4	1853.5	3 21 17.3	676.96
5	955.04	1 58 43.2	964.92
6	596.90	2 03 26.5	659.90
7	307.88	3 05 41.5	1191.10
8	2600.9	3 00 45.0	1087.20
9	1947.8	2 59 05.0	1114.90
10	1400.9	2 32 36.7	1191.10

TABLE III.

Observed and Mean Values of A and A/B.

Number of Spring.	Individual Values of A.	Mean Value of A.	Individual Values of A/B.	Mean Value of A/B.
1	11910	11900 $\pm$ 2.1	.7781	.7768 $\pm$.00036
	11900		.7776	
	11900		.7751	
	11890		.7761	
	11900		.7770	
2	12020	12008 $\pm$ 2.5	.7794	.7785 $\pm$.00017
	12010		.7789	
	12010		.7783	
	12000		.7778	
			.7779	
3		16633 $\pm$ 6.5	.7786 (<i>b</i>)	.7794 $\pm$.00024
	16660		.7798	
	16640		.7792	
	16650		.7789	
	16650		.7799	
	16660		.7794	
	16650		.7793	
	16640		.7814	
	16620		.7781	
	16590		.7776	
4	16570	16736 $\pm$ 4.5	.7806	.7865 $\pm$.00018
	16760		.7872	
	16740		.7859	
	16730		.7856	
	16730		.7871	
5	16720	92130 $\pm$ 51	.7869	.7803 $\pm$.00045
			.7864 (<i>b</i>)	
	92120		.7835	
	92090		.7794	
	92150		.7791	
	92473		.7802	
6	92130	136120 $\pm$ 260	.7795	.7716 $\pm$.00028
	91830		.7799 (<i>b</i>)	
	137000		.7715	
	135700		.7716	
	136400		.7697	
	135700		.7724	
	135700		.7723	
			.7723 (<i>b</i>)	

TABLE III.—*Continued.*
Observed and Mean Values of A and A/B.

Number of Spring.	Individual Values of A.	Mean Value of A.	Individual Values of A/B.	Mean Value of A/B.
7	752000	747600 ± 1600	.7845	$.7811 \pm .0013$
	751500		.7832	
	743900		.7814	
	743000		.7754	
8	9914	9874 ± 9	.7489	$.7463 \pm .0007$
	9873		.7437	
	9855		.7510	
	9854		.7432	
			.7496	
			.7471	
			.7427	
			.7442	
9	25100	24875 ± 66	.7448	$.7460 \pm .0009$
	24970		.7462	
	24760		.7509	
	24670		.7486	
			.7420	
			.7433	
10	65410	61742 ± 350	.7500	$.7452 \pm .0006$
	62270		.7434	
	62860		.7414	
	61430		.7468	
			.7471	
			.7449	
			.7498	
			.7419	
			.7424	
			.7437	

TABLE IV.
Computed Values of B, E, n and η .

Number and Metal of Spring.	Modulus of Bending. B	Young's Modulus. E	Modulus of Rigidity. n	Poisson's Ratio. η
		Dynes per Square Centimeter.	Dynes per Square Centimeter.	
1. Steel.	15320 ± 7.6	$(1837 \pm 0.9) \times 10^9$	$(7134 \pm 1.3) \times 10^8$	0.2873 ± 6
2. "	15420 ± 4.6	$(1951 \pm 0.6) \times 10^9$	$(7592 \pm 2.0) \times 10^8$	$.2845 \pm 3$
3. "	21340 ± 10.6	$(1933 \pm 0.9) \times 10^9$	$(7533 \pm 2.9) \times 10^8$	$.2830 \pm 4$
4. "	21280 ± 7.5	$(1899 \pm 0.7) \times 10^9$	$(7469 \pm 2.0) \times 10^8$	$.2715 \pm 3$
5. "	118100 ± 94	$(1949 \pm 1.6) \times 10^9$	$(7604 \pm 4.2) \times 10^8$	$.2816 \pm 7$
6. "	176400 ± 343	$(1939 \pm 3.8) \times 10^9$	$(7481 \pm 14.3) \times 10^8$	$.2960 \pm 5$
7. "	957100 ± 2594	$(2171 \pm 5.8) \times 10^9$	$(8479 \pm 18.1) \times 10^8$	$.2802 \pm 22$
8. Brass.	13230 ± 17	$(911 \pm 1.2) \times 10^9$	$(340 \pm 0.3) \times 10^9$	$.3400 \pm 13$
9. "	33340 ± 97	$(904 \pm 2.6) \times 10^9$	$(337 \pm 0.9) \times 10^9$	$.3405 \pm 16$
10. "	82850 ± 474	$(938 \pm 5.4) \times 10^9$	$(345 \pm 2.0) \times 10^9$	$.3419 \pm 10$

6. *Precision of the Constants Derived.*—From equation (8), in view of the small value of $\Delta\varphi/\varphi_0$, it is seen that so far as errors of observation are concerned the modulus of twist may be defined by

$$A = \frac{Ps^3 \cos^2 \alpha}{(z - z_0) \varphi_0^2}.$$

Since $\cos^2 \alpha = 1 - (z/s)^2$, and since z/s is small for each of the springs, it appears that the relative errors of A from different determinations for the same spring depend essentially on the errors of P and $(z - z_0)$ only. It may be safely assumed, however, that the errors of P , and likewise those of φ_0 and s , are negligible in comparison with those of $(z - z_0)$. Thus we conclude that

$$\frac{\Delta A}{A} = - \frac{\Delta(z - z_0)}{z - z_0},$$

or that the probable error of A is the same fraction of A as the probable error of $(z - z_0)$ is of it. We believe, therefore, that the probable errors of the mean values of A are essentially correct as given in Table III.

From similar reasoning we conclude that the probable errors of the ratios A/B given in Table III. afford trustworthy indications of the precision of the mean values of those quantities.

For computing the probable errors of B , E , n , and γ from those of the independently observed quantities A and A/B , it is essential to note the relations expressed by equations (18) to (20) in connection with the two following equations. Thus let

$$A = u, \quad \text{and} \quad \frac{A}{B} = \xi,$$

wherein u and ξ are supposed to be observed quantities. Then

$$B = \frac{u}{\xi},$$

and if ∂B , ∂u , and $\partial \xi$ denote the probable errors of B , u and ξ , respectively,

$$\left(\frac{\partial B}{B} \right)^2 = \left(\frac{\partial u}{u} \right)^2 + \left(\frac{\partial \xi}{\xi} \right)^2.$$

Hence from the values of ∂u and $\partial \xi$ given in Table III, the probable errors of B in Table IV. are derived.

From equation (18) it is seen that the probable error of n is the same fraction of n as the probable error of A is of it; and similarly from (19) the probable error of E is the same fraction of E as the probable error of B is of it.

Finally, from equation (20) it appears that the probable error of γ is equal to the probable error of the reciprocal of A/B ; and this is the same as $(B/A)^2$ times the probable error of A/B .

It will be observed from Table IV. that the precision is highest, and of about the same order, for the first four steel springs; the probable errors of A and n being about $1/3500$ part, those of B and E about $1/2000$ part, and those of γ about $1/700$ part. This higher precision is most likely due to the fact that the springs in question were the best as to uniformity, temper and finish of all the springs used. Next in order of precision are the constants for the steel springs Nos. 5 and 6. These are inferior in quality to Nos. 1-4 as regards uniformity and finish. Steel spring No. 7 is markedly different from the others in being "open-wound," of large diameter, of heavier wire than the others, and relatively short in axial length z_0 . The means at hand did not permit a sufficient elongation of this spring to attain results of higher precision for A , and the ratio A/B was also less definitely measurable than in the cases of the other springs.

The precision is relatively low, as would be expected, for the constants of the brass springs, although they are surprisingly uniform as to slope and diameter for commercial products. The well-known instability of brass as compared with steel is undoubtedly the source of this lower order of precision.

7. *Summary of Conclusions.*—From the details of the investigation given above the following conclusions appear to be established:

(a) The static theory of helical springs as expressed by equations (1) to (4), and given for the first time by Kelvin and Tait in the first edition of their *Natural Philosophy*, 1867, has been experimentally verified. Starting from this theory the above paper shows that:

(b) To a high order of approximation the force required to elongate a helical spring varies directly as the product of the elongation of the spring and the square of the secant of its slope. The exact law for finite elongations of any such spring is expressed in a simple form by equations (7), (8) and (14).

(c) When such springs are elongated by an axial pull, with no applied torsional couple, they undergo a twist about the axis of the helix. This twist increases as the pull increases up to a maximum, and thereafter decreases continuously with increasing pull.

(d) The law connecting the elongation of a spring with the applied force furnishes a ready method of finding the modulus of twist and hence the modulus of rigidity of the wire of the spring.

(e) Observations of the elongations of maximum twist and zero twist furnish independent methods of deriving the ratio of the modulus of twist to the modulus of bending.

(f) By a combination of the results furnished by (d) and (e) the modulus of bending, Young's modulus, and Poisson's ratio are readily derived.

(g) By the methods explained it is easily practicable to attain, in the case of a well-made steel spring, a precision indicated in round numbers by a probable error of one part in three thousand for the modulus of twist, of one part in two thousand for Young's modulus and of one part in five hundred for Poisson's ratio.

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